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LETTER TO THE EDITOR

Ultradiffusion on multifurcating hierarchical structures

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Abstract. Diffusion on a trifurcating hierarchical structure is studied through an exact renormalisation group procedure. The non-universal time scaling exponent of the autocorrelation function is obtained and compared with that of the simplest 1D ultradiffusion model. When a biasing term is introduced, it is found that the transition from power law to exponential decay of the autocorrelation function is independent of the furcating number of the hierarchy. A general Z-furcating hierarchical system is also discussed briefly.

It is known that hierarchical structures appear in different physical contexts ranging from molecular diffusion (Austin *et al* 1975) to spin glasses (Sompolinsky 1981) and computing architectures (Huberman and Hogg 1984). The earliest model proposed by Huberman and Kerzberg (1985) for relaxation in hierarchical structures consists in random walks of a particle in a one-dimensional chain with a regular uniformly bifurcating hierarchical array of barriers. It is termed the 1D ultradiffusion model due to the characteristic ultrametric topology of the system (Bourbaki 1966). Diffusion on such a simple hierarchical structure has been exactly solved within a renormalisation group treatment (Maritan and Stella 1986) and the anomalous exponent x for the long-time behaviour of the autocorrelation function $P_0(t) \sim t^{-x/2}$ was found to be

$$x = 2 \ln 2/\ln[2(2w_1^* + w_0)/w_1^*] \tag{1}$$

where w_1^* characterises the line of fixed points to which the initial barrier hierarchy $\{w_n\}$ is attracted. In particular if $w_j = R^j$ ($j = 0, 1, \ldots$) one obtains

$$x = \begin{cases} 2 \ln 2/(\ln 2 - \ln R) & 0 < R < \frac{1}{2} \\ 1 & \frac{1}{2} < R < 1 \end{cases} \tag{2}$$

and a dynamical phase transition from ordinary to anomalous diffusion is found at $R_c = \frac{1}{2}$.

Further, Ceccatto and Riera (1986) considered a generalisation of the above model, i.e. the 1D ultradiffusion model with a biasing term η. For a suitable value ($\eta = w_0$) of the biasing force, they found a new transition in the relaxational behaviour of the autocorrelation function, from power law to exponential decay.

In this letter, we study diffusion on a one-dimensional trifurcating hierarchical structure as shown in figure 1. The corresponding anomalous exponent is obtained by an exact renormalisation method and a general expression for a z-furcating hierarchical structure is suggested. We also study the biased diffusion on this structure and

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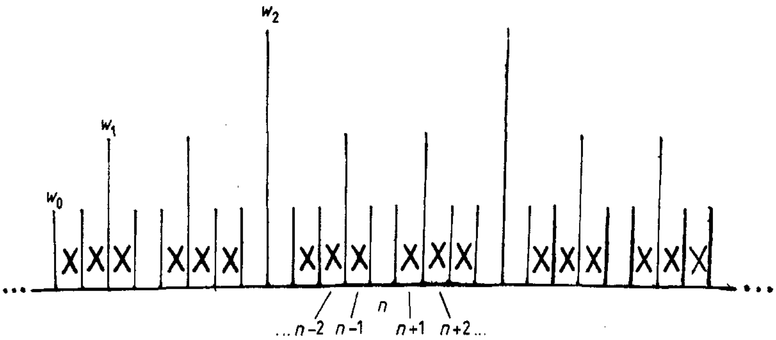

Figure 1. Trifurcating hierarchical barrier structure. The cells with crosses are decimated in the renormalisation group procedure.

show that the corresponding new transition from power law to exponential decay of $P_0(t)$ is just also at $\eta = w_0$, i.e. it does not depend on the furcating ~~number of the~~ hierarchy. We conjecture that this conclusion is tenable for the general z-furcating hierarchical system.

Let $P_m(t)$ be the probability of finding the particle at time t at cell m. Its Laplace transform $\tilde{P}_m(\omega)$ satisfies the following equation:

$$\omega\tilde{P}_m = w_{m+1,m}(\tilde{P}_{m+1} - \tilde{P}_m) + w_{m-1,m}(\tilde{P}_{m-1} - \tilde{P}_m) + \delta_{m,0} \tag{3}$$

where $w_{m\pm1,m}$ is the hopping rate between the nearest-neighbour cells $m \pm 1$ and m. It takes the appropriate value associated with the corresponding barrier (the taller the barrier, the smaller W_j). $\delta_{m,0}$ means that the particle is supposed to start its random walk from the cell 0.

Performing an exact renormalisation group decimation procedure which eliminates the cells marked by crosses in figure 1, we obtained a new system of the same form as (3) with the rescaled parameters which are given by the following recursion relations:

$$w'_j = \frac{w_0 + 3w_1}{w_1} w_{j+1} \tag{4}$$

$$\tilde{P}'_n = \frac{w_1}{w_0 + 3w_1} \tilde{P}_n \tag{5}$$

$$\begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix} = \begin{pmatrix} (w_0+5w_1)/w_1 & (2w_0+8w_1)/w_1 \\ (w_0+2w_1)/w_1 & (2w_0+5w_1)/w_1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{6}$$

Note that we have two effective eigenvalues in equation (3), i.e. ω'_1 and ω'_2 for the central cells $n+3k$ ($k = 0, \pm1, \pm2, \ldots$) and the rest respectively, while the initial values $\omega_1 = \omega_2 = \omega$. In deriving relations (4)–(6) we have used the condition $w'_0 = w_0$ to fix the timescale and considered only the $\omega \to 0$ limit to obtain the leading dynamical scaling behaviour of the autocorrelation function $P_0(t)$.

The transforming matrix in equation (6) has the maximum eigenvalue

$$\lambda_{\max} = 3(3w_1 + w_0)/w_1. \tag{7}$$

Taking into account (4)–(7) and paying attention to the inverse Laplace transform of $\tilde{P}_0(\omega)$ one obtains

$$x = 2\ln 3/\ln[3(3w_1^* + w_0)/w_1^*] \tag{8}$$

where w_1^* characterises a whole line of fixed points to which the initial $\{w_n\}$ is attracted under iteration of (4). The exponent varies continuously between $x=0$ (trapping) and $x=1$ (normal diffusion) with w_1^* ranging from 0 to $+\infty$. For the particular set of transition rates $w_j = R^j$ ($j=0, 1, \ldots$) one has

$$x=\begin{cases} 2\ln 3/(\ln 3-\ln R) & 0<R<\frac{1}{3} \\ 1 & \frac{1}{3}<R<1. \end{cases} \tag{9}$$

It is shown that the crossover for the dynamical phase transition from ordinary to anomalous diffusion is at $R_c=\frac{1}{3}$. Comparing (8) and (9) with (1) and (2) respectively one may suggest the following relations:

$$x=2\ln z/\ln[z(zw_1^*+w_0)/w_1^*] \tag{10}$$

and

$$x=\begin{cases} 2\ln z/(ln\, z-\ln R) & 0<R<1/z \\ 1 & 1/z<R<1 \end{cases} \tag{11}$$

for a corresponding general z-furcating hierarchical structure. Equation (11) gives the transition at $R_c=1/z$, which has a direct relation to the furcating number of the hierarchy.

Now introducing a biasing term η to the above trifurcating hierarchical system one has

$$\begin{aligned} \omega\tilde{P}_m &= \delta_{m,0}+(w_{m-1,m}+\eta)\tilde{P}_{m-1}+(w_{m+1,m}-\eta)\tilde{P}_{m+1}-(w_{m,m+1}+w_{m,m-1})\tilde{P}_m \\ &= \delta_{m,0}+w_{m-1,m}(\tilde{P}_{m-1}-\tilde{P}_m)+w_{m,m+1}(\tilde{P}_{m+1}-\tilde{P}_m)+\eta(\tilde{P}_{m-1}-\tilde{P}_{m+1}) \end{aligned} \tag{12}$$

where $\eta\leqslant\min\{w_n\}$ so that the model seems reasonable. Also by the decimation technique the following recursion relations can be obtained

$$\begin{gathered} w_j'=\Box w_{j+1} \qquad \eta'=\Box\eta \qquad P_n'=P_n/\Box \\ \begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix}=\begin{pmatrix}\Phi_{11} & \Phi_{12}\\ \Phi_{21} & \Phi_{22}\end{pmatrix}\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix} \end{gathered} \tag{13}$$

where

$$\begin{aligned} \Box &= w_0(w_0^3+3w_0^2w_1+3w_0\eta^2+w_1\eta^2)/\Delta \\ \Phi_{11} &= w_0(w_0^3+5w_0^2w_1+7w_0\eta^2+3w_1\eta^2)/\Delta \\ \Phi_{12} &= 2w_0^2(w_0^2+4w_0w_1+3\eta^2)/\Delta \\ \Phi_{21} &= w_0(w_0^3+2w_0^2w_1+w_0\eta^2-w_0^2\eta-2w_1w_0\eta-\eta^3)/\Delta \\ \Phi_{22} &= w_0(2w_0^3+5w_0^2w_1+6w_0\eta^2-3w_0^2\eta-4w_1w_0\eta+3w_1\eta^2-\eta^3)/\Delta \\ \Delta &= w_0^3w_1+3w_0^3\eta^2+3w_0w_1\eta^2+\eta^4. \end{aligned} \tag{14}$$

In (13) and (14) we have also put $w_0'=w_0$ to fix the unit of time and removed terms of $O(\omega)$ in $\Box$, Δ and Φ_{ij} ($i,j=1,2$) in order to obtain the leading singular behaviour of $P_0(t)$ for $t\to\infty$.

It can be found that the recursion $\eta'=\Box\eta$ has the following fixed points:

$$\eta^*=\begin{cases} 0 & \min\{w_n\}<w_0 \\ 0,\ w_0 & \min\{w_n\}=w_0. \end{cases} \tag{15}$$

The fixed point $\eta^* = 0$ corresponds to the above unbiased trifurcating model, and the fixed point $\eta^* = w_0$ leads to a new transition from potential to exponential decay, which can be solved as

$$P_0(t) = \tfrac{1}{2}(e^{-\gamma_1 t} + e^{-\gamma_2 t}) \tag{16}$$

where

$$\gamma_{1,2} = (w_{0,-1} + w_{0,1}) \pm |w_{0,-1}^2 - w_{0,1}^2|^{1/2}. \tag{17}$$

Comparing our results with those obtained by Ceccatto and Riera (1986), it is thus obvious that this new transition from power law to simple exponential decay of $P_0(t)$ does not depend on the furcating number of the hierarchy. It would appear reasonable to expect that this may be valid for a general z-furcating hierarchical system.

To conclude, we have investigated diffusion on the trifurcating hierarchical barrier structure using the exact renormalisation decimation technique and obtained the anomalous diffusion exponent for the autocorrelation function. The corresponding expressions for the general z-furcating hierarchical system were also suggested. On one hand the anomalous diffusion exponent x and the phase transition point from ordinary to anomalous diffusion in the case of a particular hierarchy depend on the furcating number of the hierarchy; on the other hand, it is shown that the transition from power law to exponential decay of the autocorrelation function, which results from the biased diffusion, is independent of the furcating number of the hierarchy.

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